Gaussian wave packet solution of the Schrodinger equation in the presence of a time-dependent linear potential

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Abstract

We argue that the way to get the general solution of a Schrodinger equation in the presence of a time-dependent linear potential based on the Lewis-Riesenfeld framework is to use a Hermitian linear invariant operator. We demonstrate that the linear invariant proposed in p and q is an Hermitian operator which has the Gaussian wave packet as its eigenfunction.

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The time evolution of a quantum system subject to a spatially uniform, time-dependent force has attracted considerable interest lately. The exact propagator for this system has long been known [1], as have a set of exact solutions (the Volkov solutions) [2-3]. Recent work [4–13] focuses further on exact solutions and their properties. First, Guedes [5] obtained, by means of the invariant operator method introduced by Lewis and Riesenfeld (L-R) [14], a special solution for the time-dependent linear potential. The idea is that any operator satisfying the quantum Liouville-von Neumann equation provides its eigenstate as a solution of the time-dependent Schrodinger equation up to a time dependent phase factor. Later on, Feng [6] followed a method based on spatiotemporal transformations of the Schrodinger equation to get the plane-wave-type and the Airy-packet solutions where the one in Ref. [5] constitutes a particular case which corresponds to the so-called "standing" particle case in a linear potential [6].

However, Bekkar et al. [8] pointed out that the Airy-packet solution is in fact only a superposition of the plane-wave-type solution. In his Comment [7], Bauer explained that the solution found by Guedes [5] is simply a special case of the Volkov solution, with a zero wave vector k, to the time dependent Schrodinger equation describing a nonrelativistic charged particle moving in an electromagnetic field. Dunkel and Trigger [10] considered the initial minimum-uncertainty Gaussian for a sinusoidally time-dependent linear potential. Bowman [11] investigated the time evolution of the general quantum state for the time-dependent linear system, which was shown to be that of a free-particle state, plus an overall motion arising from the classical force . Sang Pyo Kim [12] has shown that a charged free particle in a constant and/or oscillating electric field has a bounded Gaussian wave packet, which is a coherent state of a one-parameter and time-dependent ground state for the free-particle Hamiltonian. From the test-function method, Gengbiao Lu et al.[13] constructed an exact n wave-packet-train (n GWPT) solution , whose center moves acceleratively along the corresponding classical trajectory.

Very recently, Luan et al. [9] have reexamined the linear invariant proposed by Guedes [5] and indicated that besides the solutions described in Refs. [5-8], a Gaussian wave packet eigenfunction solution can naturally be derived from the LR method if a non-Hermitian linear invariant is used, and they argue that this solution was ruled out before, because the authors in Refs. [5-8] assumed in advance the linear L-R invariant $I(t) = A(t)p + B(t)x + C(t)$ be a Hermitian operator.

The authors [9] stated that in Ref.[5, 8] , the Gaussian wave packet eigenfunction solution was ruled out , because the linear LR invariant is supposed to be Hermitian operator, and the B parameter was set to zero for hermiticity reason. In fact this is not the reason for setting $B = 0$, in Ref. [8], it was meant to justify the $B = 0$ imposed in Ref. [5]. We will show and we will correct in the present paper, that if B is different from zero ($B \neq 0$), we still obtain Gaussian wave packet (GWP) solutions. In fact, we could find all the results using the Hermitian invariant operator linear in p and q with any conditions imposed to get physically acceptable solutions as was done in Luan et al. paper [9]. In the latter approach, the authors obtained solutions labelled by complex parameters and complex eigenvalues, this is ambiguous.

We recall that according to the theory of Lewis and Riesenfeld [14], an invariant is an operator that must necessarily satisfy three requirements: 1. It is Hermitian. 2. It satisfies the Von Neumann equation. 3. Its eigenvalues are real and time-independent (we signal that the Hermiticity of the invariant is one of the essential conditions which makes the eigenvalues of the invariant time-independent). Furthermore, any invariant satisfying these three requirements leads to a complete set of solutions of the corresponding Schrodinger equation. So, a conventional solution is constructed as a linear combination of these solutions.

The problem is to find the solutions of the Schrodinger equation,

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = H\ \psi(x,t) \tag{1}$$

for the Hamiltonian

$$H\left(x,p,t\right) = \frac{1}{2m}p^2 - F(t)x, \tag{2}$$

where $F(t)$ is a time-dependent function.

According to the theory of Lewis and Riesenfeld [14], a solution of the Schro¨dinger equation with a time-dependent Hamiltonian is easily found if a nontrivial Hermitian operator $I(t)$ exists and satisfies the invariant equation

$$\frac{dI}{dt} = \frac{\partial I}{\partial t} + \frac{1}{i\hbar}\left[I,H\right] = 0. \tag{3}$$

Indeed, this equation is equivalent to saying that if $\varphi_\lambda(x,t)$ is an eigenfunction of $I(t)$ with a time-independent eigenvalue λ, we can find a solution of the Schrodinger equation in the form $\psi_\lambda(x,t) = \exp\left[i\alpha_\lambda\left(t\right)\right]\varphi_\lambda(x,t)$ where $\alpha_\lambda\left(t\right)$ satisfies the eigenvalue equation for the Schrodinger operator,

$$\hbar\dot{\alpha}_\lambda\left(t\right)\varphi_\lambda = i\left[\hbar\frac{\partial}{\partial t} - H\right]\varphi_\lambda. \tag{4}$$

It turns out that the time-dependent invariant operator takes the linear form [9]

$$I(t) = A(t)p + B(t)x + C(t). \tag{5}$$

The invariant equation is satisfied if the time-dependent coefficients are such that

$$A\left(t\right) = A_0 - \frac{B_0}{m}t, \qquad B(t) = B_0, \tag{6}$$

$$C\left(t\right) = C_0 - A_0\int_0^t F(\tau)d\tau + \frac{B_0}{m}\int_0^t F(\tau)\tau d\tau, \tag{7}$$

where A_0, B_0, C_0 are arbitrary real constants.

The eigenstates of $I(t)$ corresponding to time-independent eigenvalues are the solutions of the equation

$$I(t)\varphi_\lambda(x,t) = \lambda\ \varphi_\lambda(x,t). \tag{8}$$

It is easy to see that the solutions of Eq. (8) are of the form

$$\varphi_\lambda(x,t) = \exp\left\{\frac{i}{\hbar}\left[\frac{2\left(\lambda - C\left(t\right)\right)x - B_0x^2}{2A\left(t\right)}\right]\right\}. \tag{9}$$

Substituting Eq. (9) into Eq. (4) and accomplishing the integration, we obtain

$$\begin{aligned}\alpha_\lambda(t) &= \alpha_\lambda(0) - \int_0^t \left\{ \frac{(\lambda - C(\tau))^2}{2m\hbar A(\tau)^2} + \frac{iB_0}{2mA(\tau)} \right\} d\tau \\ &= \alpha_\lambda(0) - \int_0^t \left\{ \frac{(\lambda - C(\tau))^2}{2m\hbar A(\tau)^2} \right\} d\tau - i\mathrm{Ln}\left(\sqrt{\frac{A_0}{A(t)}} \right), \end{aligned} \tag{10}$$

note that the logarithmic term goes downstairs as a time-dependent normalization factor in $\psi_\lambda(x,t)$.

Therefore the physical orthogonal wave functions $\psi_\lambda(x,t)$ solutions of the Schrodinger equation (1) are given by

$$\psi_\lambda(x,t) = \sqrt{\frac{A_0}{A(t)}} \exp\left\{ -i \int_0^t \frac{(\lambda - C(t'))^2}{2\hbar m A(t')^2} dt' + \right\} \exp\left\{ \frac{i}{\hbar} \left[\frac{2(\lambda - C(t))x - B_0 x^2}{2A(t)} \right] \right\}. \tag{11}$$

It is easy to verify that

$$\langle \psi_\lambda \mid \psi_{\lambda'} \rangle = \delta(\lambda - \lambda'). \tag{12}$$

Furethermore, the evolution of the general Schrodinger state can be writen as

$$\Psi(x,t) = \int_{-\infty}^{+\infty} g(\lambda)\, \psi_\lambda(x,t)\, d\lambda, \tag{13}$$

where $g(\lambda)$ is a weight function which determines the state of the system such that $\Psi(x,t)$ is square integrable i.e. $\int \mid \Psi(x,t) \mid dx$ is time-independent finite constant.

In Ref. [8] , it has been shown that the choice

$$g(\lambda) = \exp\left(\frac{i\lambda^3}{3} \right) \tag{14}$$

leads to Airy function solutions.

Here we shall show that one obtains a general Gaussian wave-packet solution by choosing weight function as a gaussian too,

$$g(\lambda) = \sqrt{\frac{\sqrt{a}}{\hbar A_0 \pi \sqrt{2\pi}}} \exp\left(-a\lambda^2\right), \tag{15}$$

where a is a positive real constant.

Substituting Eq. (11) and (15) into Eq. (13) and accomplishing the integration, we obtain the normalized Gaussian solution as

$$\begin{aligned}\Psi(x,t) &= \sqrt{\frac{\sqrt{a}}{\hbar A(t)\sqrt{2\pi}\left(a + i\int_0^t \frac{1}{2\hbar m A(t')^2} dt' \right)}} \exp\left\{ -i\int_0^t \frac{C(t')^2}{2\hbar m A(t')^2} dt' \right\} \\ &\quad \exp\left\{ \frac{i}{\hbar}\left[\frac{-2C(t)x - B_0 x^2}{2A(t)} \right] \right\} \exp\left\{ \frac{-\left[\int_0^t \frac{C(t')}{\hbar m A(t')^2} dt' + \frac{x}{\hbar A(t)} \right]^2}{\left(4a + i\int_0^t \frac{2}{\hbar m A(t')^2} dt' \right)} \right\} \end{aligned} \tag{16}$$

Let us now evaluate the mean value of x, p and the quantum coordinate and momentum fluctuation in the state $\Psi(x,t)$. After some algebra we find that

$$\langle x \rangle = \langle \Psi(t) | x | \Psi(t) \rangle = -A(t) \int_0^t \frac{C(t')}{mA(t')^2} dt', \tag{17}$$

which is nothing but the classical position $x_c(t)$, and

$$\langle p\rangle = \langle \Psi(t)|p|\Psi(t)\rangle = \frac{-C(t)}{A(t)} + \int_0^t \frac{B_0 C(t')}{mA(t')^2}dt' \tag{18}$$

is a classical momentum $p_c(t)$. The position uncertainty

$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \hbar A(t)\sqrt{\frac{\left(a^2 + \left(\int_0^t \frac{1}{2\hbar mA(t')^2}dt'\right)^2\right)}{a}}, \tag{19}$$

and the momentum uncertainty

$$\begin{aligned}\Delta p &= \sqrt{\langle p^2\rangle - \langle p\rangle^2} \\ &= \frac{1}{\Delta x}\sqrt{\frac{\hbar^2}{4} + \left[\int_0^t \frac{1}{4a\ mA(t')^2}dt' - \frac{\hbar^2 B_0 A(t)}{a}\left(a^2 + \left(\int_0^t \frac{1}{2\hbar mA(t')^2}dt'\right)^2\right)\right]^2},\end{aligned} \tag{20}$$

leads to the uncertainty relation

$$\Delta p\Delta x = \sqrt{\frac{\hbar^2}{4} + \left[\int_0^t \frac{1}{4a\ mA(t')^2}dt' - \frac{\hbar^2 B_0 A(t)}{a}\left(a^2 + \left(\int_0^t \frac{1}{2\hbar mA(t')^2}dt'\right)^2\right)\right]^2} \geq \frac{\hbar}{2}. \tag{21}$$

We can rewrite (16) as follows

$$\begin{aligned}\Psi(x,t) &= \sqrt{\frac{\hbar A(t)\left(a - i\int_0^t \frac{1}{2\hbar mA(t')^2}dt'\right)}{\sqrt{2\pi a}\Delta x^2}}\exp\left\{-i\int_0^t \frac{C(t')^2}{2\hbar mA(t')^2}dt'\right\} \\ &\quad \exp\left\{\frac{i}{\hbar}\left[\frac{-2C(t)x - B_0 x^2}{2A(t)}\right]\right\}\exp\left\{\frac{-\left(a - i\int_0^t \frac{1}{2\hbar mA(t')^2}dt'\right)[x - \langle x\rangle]^2}{4a\Delta x^2}\right\}.\end{aligned} \tag{22}$$

Morever, the time-dependent probability density associated with this Gaussian wave packet is Gaussian for all times

$$|\Psi(x,t)|^2 = \frac{1}{\sqrt{2\pi}\Delta x}\exp - \left\{\frac{(x - \langle x\rangle)^2}{2\Delta x^2}\right\}, \tag{23}$$

we see that Δx represents the width of the wave packet at time t. It is also readily verified tat the time-dependent probability density is conserved

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = 1 \tag{24}$$

Equations (22) and (23) describe a Gaussian wave packet that is centered at $x = \langle x\rangle$ whose width $\Delta x(t)$ varies with time. So, during time t, the packet's center has moved from $x = 0$ to $x = -A(t)\int_0^t \frac{C(t')}{mA(t')^2}dt'$ and its width has expanded from $\Delta x_0 = \hbar A_0\sqrt{a}$ to $\Delta x(t) = \Delta x_0 \frac{A(t)}{A_0}\sqrt{1 + \left(\int_0^t \frac{1}{2\hbar maA(t')^2}dt'\right)^2}$. The wave packet therefore undergoes a distortion; although it remains Gaussian, its width broadens with time whereas its height, $\frac{1}{\sqrt{2\pi}\Delta x}$, decreases with time. Further, it should be noted that the width of the Gaussian packet does not depend on the external force $F(t)$. Thus the shape of the wave packet is not changed by the external force. This means that the external force $F(t)$ acts uniformly in the wave packet.

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